Pergamon

Int. J. Solids Structures Vol. 33, No. 25, pp. 3649–3671, 1996

0020-7683/96 $15.00 + .00

0020-7683(95)00208-1

ON ASYMPTOTICALLY CORRECT LINEAR LAMINATED PLATE THEORY*

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(*Received* 30 *March* 1995; *in revised form* 30 *August* 1995)

Abstract—The focus of this paper is the development of asymptotically correct theories for laminated plates, the material properties of which vary through the thickness and for which each lamina is orthotropic. This work is based on the variational-asymptotical method, a mathematical technique by which the three-dimensional analysis of plate deformation can be split into two separate analyses: a one-dimensional through-the-thickness analysis and a two-dimensional "plate" analysis. The through-the-thickness analysis includes elastic constants for use in the plate theory and approximate closed-form recovering relations for all three-dimensional field variables expressed in terms of plate variables. In general, the specific type of plate theory that results from this procedure is determined by the procedure itself. However, in this paper only "Reissner-like" plate theories are considered, often called first-order shear deformation theories. This paper makes three main contributions: first it is shown that construction of an asymptotically correct Reissner-like theory for laminated plates of the type considered is not possible in general. Second, a new point of view on the variational-asymptotical method is presented, leading to an optimization procedure that permits a derived theory to be as close to asymptotical correctness as possible. Third, numerical results from such an optimum Reissner-like theory are presented. These results include comparisons of plate displacement as well as of three-dimensional field variables and are the best of all extant Reissner-like theories. Indeed, they even surpass results from theories that carry many more generalized displacement variables.

NOMENCLATURE

i, j, k	indices of range 1, 2, 3
$\alpha, \beta, \gamma, \delta, \mu, \nu, \eta$	the indices of range 1, 2
δ_α^β	the Kronecker's δ-symbol
$\bullet : \bullet$	the contracting operation; $C^{\alpha\beta}A_{\alpha\beta} \equiv C : A$
$\bullet_{,\alpha}$	the derivative with respect to x_α; $A_{,\alpha} \equiv \partial A / \partial x_\alpha$
$\bullet_{\underline{\alpha\beta}}$	underlined indices denote the operation of the symmetrization: $\bullet_{\underline{\alpha\beta}} \equiv \frac{1}{2}(\bullet_{\alpha\beta} + \bullet_{\beta\alpha})$
$\langle \bullet \rangle$	the averaging operation, $\int_{-1/2}^{+1/2} \bullet \, d\zeta \equiv \langle \bullet \rangle$
h	the plate thickness
l	the characteristic length of the variables
h/l	the small parameter
x_α	the plate middle plane coordinates
x_i	the 3-D space coordinates $x_i \equiv \{x_\alpha, x_3\}$
ζ	dimensionless through-the-thickness coordinate $\zeta \equiv x_3/h$
$u_i(x_j)$	components of 3-D displacement vector
$v_i(x_\alpha)$	components of plate displacement vector
θ_α	rotations for the Reissner-like theory
ε	the 3-D strains
ε	the small parameter characterizing plate deformation; also, the set of 2-D plate strains
$A_{\alpha\beta}$	the 2-D measure of the in-plane plate deformation, $A_{\alpha\beta} \equiv v_{\underline{\alpha,\beta}}$
$B_{\alpha\beta}$	the 2-D measure of the bending plate deformation for the classical plate theory, $B_{\alpha\beta} \equiv -hv_{3,\alpha\beta}$
$K_{\alpha\beta}$	the 2-D measure of the bending plate deformation for the Reissner-like theory, $K_{\alpha\beta} \equiv h\theta_{\underline{\alpha,\beta}}$
γ_α	the shear deformation for the Reissner-like theory
$\mathcal{I}_c$	the special notation: $\mathcal{I}_e \equiv 1$ and $\mathcal{I}_b \equiv \zeta$
A^a	the pair of $A \equiv A^e$ and $K \equiv A^b$; a (sometimes c) is a quasi-index that may have two values—e (corresponds to in-plane deformation) and b (corresponds to bending).

* Presented at the 36th Structures, Structural Dynamics and Materials Conference, New Orleans, Louisiana, 10–12 April 1995.

1. INTRODUCTION

For engineering structures, laminated composite materials provide excellent opportunities for structural simplicity as well as elastic couplings for potential optimization of design criteria. Although plates made of such materials have been used for some time in a variety of engineering applications, simple and efficient methods for analyzing plates with anisotropy and nonhomogeneity are still needed. This is partly due to rapid changes taking place in manufacturing technology for composite materials and partly to ever-increasing demands for accuracy and efficiency. The balance of accuracy and simplicity inherent in Reissner-like plate theories makes them quite desirable for engineering analysis. Therefore, the intent of this paper is to present a means to obtain the best possible Reissner-like theory for laminated plates.

1.1. *Background*

Much of what is done in engineering analysis of laminated plates is based on classical plate theory (CPT) which, although adequate for many engineering applications, has well known limitations due to the Kirchhoff hypothesis. Many attempts have been made to improve classical theory by taking into account non-classical behavior such as transverse shear deformation and transverse normal stresses. From the time laminated fiber-reinforced composites were first introduced, numerous works have been published, the objectives of which include the improvement of CPT for laminated plate applications. This subject is discussed at length in review papers by Librescu and Reddy (1989); Noor and Burton (1989).

There are two main classes of methods for improving plate theory found in the literature: (1) Power Series Methods: expansion of the displacement field variables into higher-order power series in the thickness coordinate; and (2) Layerwise Variables Methods: incorporation of separate sets of displacement variables for each layer. Both of these methods have known shortcomings. For example, no power series expansion can possibly render accurate results for quantities which may possess discontinuities, such as certain components of strain and stress in laminated plates. The layerwise variables methods rely on a significant increase in the number of unknowns, a number which depends on the number of layers in the plate.

A third method has received some attention in recent years, which involves an assumed displacement field with discontinuities allowed in through-the-thickness derivatives (see Murakami (1986), Sciuva (1986), Cho (1991)). There is no question that this method yields excellent results in some cases, but it lacks a systematic basis for choosing the displacement functions, and it does not yield an asymptotically correct result in general.

Atilgan and Hodges (1992) took a different approach that neither involves power-series expansion through the plate thickness nor layerwise unknowns. Rather, the three-dimensional (3-D) energy of a laminated plate was approximated following the variational-asymptotical methodology of Berdichevsky (1979). Normally asymptotical methods are employed for analytical developments, but here such a method was used in a sort of semi-analytical approach. Namely, the theory leads to a Reissner-like plate theory, along with a set of elastic constants; it also provides a set of recovering relations from which approximate 3-D displacement, strain, and stress fields can be determined once the plate equations are solved. The plate equations can be solved by any method desired, such as a 2-D finite element method. Their analysis was restricted to laminated plates for which each lamina exhibits monoclinic material symmetry about its middle surface. The first approximation is asymptotically correct for this case and coincides with CPT. The second approximation is problematic because of a certain interaction term in the strain energy. The theory is Reissner-like when this term is simply removed but asymptotically correct only when it is actually equal to zero. For example, when each element of the reduced stiffness matrix $\bar{Q}$ (see Jones (1975)) is constant through the thickness of the entire plate, the resulting theory is then Reissner-like and asymptotically correct. No method was given that would make this term vanish rigorously for the general class of plates being analyzed; it was neglected in order to provide a Reissner-like theory. Although the resulting theory is thus not asymptotically correct for the class of plates under consideration, it does provide reasonably accurate results (see Hodges *et al.* (1992)).

A similar, but somewhat improved, approach was undertaken by Hodges *et al.* (1992) and (1993), in which the estimation procedure of Atilgan and Hodges (1992) was slightly modified to include transverse shear terms in the first approximation. Plates with cross-ply stacking sequences under cylindrical bending were taken as example problems by Hodges *et al.* (1992). In a later extension of this work (Hodges *et al.* (1993)), plates with arbitrary stacking sequences undergoing cylindrical bending were taken as example problems. The material configurations of these latter plates are not as simple as those of bi-directional plates, because of the influence of the coupling of transverse shear terms. The distributions of 3-D displacement, strain, and stress were investigated for both cases by comparing the corresponding 3-D exact elasticity solution (see Pagano (1969, 1970)). The theory of Hodges *et al.* (1992) and Hodges *et al.* (1993), termed the "neo-classical" plate theory (NCPT), was shown to be more accurate than CPT for thick, laminated plates; also, it was shown to yield results which are somewhat better than those of the theory of Atilgan and Hodges (1992). Still, there were results reported for which the correlation of NCPT with the exact solution is not good. For example, when shear coupling exists, NCPT shows significantly better correlation with the exact solution than for the bi-directional cases. Also, it was necessary to integrate the 3-D equilibrium equations in order to obtain accurate transverse stresses and strains. Although it is not discussed by Hodges *et al.* (1992) and Hodges *et al.* (1993), it is important to note that extending these approaches to higher approximations requires interaction terms to vanish that are analogous to the one neglected by Atilgan and Hodges (1992).

An attempt to avoid difficulties with these interaction terms was made by Lee *et al.* (1993), where a refined theory based on using eigenmodes through the thickness as "new degrees of freedom" was developed. (While it is true that a plate has an infinite number of degrees of freedom, each normal line element has a finite number. In this sense, classical theory has 3, while Reissner-like theories have 5. The refined theory proposed by Lee *et al.* (1993) has $3+n$, where n is the number of new degrees of freedom.) A proper choice of the modal functions for the new degrees of freedom was supposed to compensate for the lack of asymptotical correctness, and eigenfunctions were proposed as a choice (see motivation in Lee *et al.* (1993).) The theory proposed therein is asymptotically correct with respect to each individual degree of freedom, but the order of each degree of freedom was considered to be independent of the orders of the others. The reason for this is that the order of each degree of freedom in fact depends on applied external forces, which can be arbitrary. However, the order of each degree of freedom becomes dependent on that of the others when the external forces are null, for example; thus, this theory is not, and cannot be, asymptotically correct in full sense of the previous works. It is possible that the eigenfunction theory cannot be finished without non-trivial short wavelength extrapolation. (See, for example, analogous work in beam theory of Cesnik *et al.* (1994).) Investigation of this approach has not yet been completed.

Were the problem of the interaction terms overcome, the already good accuracy of the approaches by Atilgan and Hodges (1992) and Hodges *et al.* (1993) would be enhanced significantly. The resulting theory would provide a very simple means to accurately analyze laminated plates—far simpler for given accuracy than any of the higher-order and layerwise theories. The potential for this methodology to yield an asymptotically correct theory provides compelling motivation to attempt again to deal with the troublesome interaction term that was neglected by Atilgan and Hodges (1992).

1.2. *Present approach*

A method for dealing with these terms has now been developed. This method, the attendant theory, and observations on asymptotical correctness, are the subjects of the present paper. We will show that the issue of asymptotical correctness is the most important one. The closer a theory is to being asymptotically correct, the better it is. On the other hand, the simplicity and physical appeal of the Reissner-like theory cannot be denied. These are the main motivations for attempting to construct an asymptotically correct, Reissner-like theory.

A linear theory for laminated plates, each layer of which is orthotropic, is presented. However, there appear to be no obstacles to develop a nonlinear theory for plates with arbitrary physical properties. In this paper, we use some forms of tensor notation that are not in common use but are suitable for the purpose of the present investigation. The approach to be presented is actually equivalent to the variational-asymptotical method, combined with a change of variables implemented in a number of papers Berdichevsky (1979), Berdichevsky and Starosel'skii (1983) and Le (1986). The essence of this new approach is that we look directly for the recovering relations that keep the 2-D energy Reissner-like and still give an asymptotically correct theory.

We begin with the 3-D formulation of the problem. Next, the notion of asymptotical correctness is defined. Classical theory is then presented, along with asymptotically correct recovering relations. Reissner theory is then presented, first for homogeneous, isotropic plates and then in its general form. Finally, we present some numerical results.

2. ORIGINAL 3-D FORMULATION OF THE PROBLEM

In this paper we will consider only linear plate theory. Let a Cartesian system of coordinates, x_i, be chosen in such a way that $x_\alpha: \{x_1, x_2\}$ denotes lengths along orthogonal straight lines in the mid-surface of the undeformed plate, and $x_3 \equiv h\zeta$ is the distance in the normal direction, where $-\frac{1}{2} \leqslant \zeta \leqslant \frac{1}{2}$ and h is the thickness of the plate. Throughout the analysis, Greek indices assume values 1 or 2; Latin indices assume values 1, 2, and 3; and repeated indices are summed over their ranges. Note that the variable ζ may not be regarded as an index.

Asymptotical methods imply the existence of small parameters. In a linear theory one supposes the strain to be small. In addition to this, we consider the thickness of the plate h as a small quantity. The deformed state of plate is described by three spatial displacements, u_i, as functions of x_i. In order to explicitly introduce h into the equations, we replace the coordinate x_3 by $h\zeta$ and consider the displacement as a function of ζ, h and the plate coordinates x_α:

$$u_i(x_i) = u_i(x_\alpha, \zeta, h). \tag{1}$$

In this notation, the strain measures are

$$\begin{aligned} 2\varepsilon_{\alpha\beta} &= u_{\alpha,\beta}(x_\alpha, \zeta, h) + u_{\beta,\alpha}(x_\alpha, \zeta, h) \\ 2\varepsilon_{\alpha 3} &= \frac{1}{h} u_{\alpha,\zeta}(x_\alpha, \zeta, h) + u_{3,\alpha}(x_\alpha, \zeta, h) \\ \varepsilon_{33} &= \frac{1}{h} u_{3,\zeta}(x_\alpha, \zeta, h). \end{aligned} \tag{2}$$

Here and below $\bullet_{,\alpha} \equiv \bullet_{,x_\alpha} \equiv \partial \bullet / \partial x_\alpha$ and $\bullet_{,\zeta} \equiv \partial \bullet / \partial \zeta$.

The strain energy density of a laminated plate, each lamina of which is made of orthotropic material, is written as

$$U = \tfrac{1}{2} D_\perp (\varepsilon_{33} + C^{\gamma\delta}\varepsilon_{\gamma\delta})^2 + \tfrac{1}{2} D_\angle^{\alpha\beta}(2\varepsilon_{\alpha 3})(2\varepsilon_{\beta 3}) + \tfrac{1}{2} D^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} \tag{3}$$

where $D_\perp$, $D_\angle^{\alpha\beta}$ and $D^{\alpha\beta\gamma\delta}$ are the dimensional material constants for transverse, shear, and in-plane energies, respectively; $C^{\gamma\delta}$ are the generalized dimensionless 2-D Poisson's ratios. This way of writing the energy density, taken from Berdichevsky (1979), is unusual, but it is a natural way to facilitate development of a plate theory. All the material constants may be functions of the transverse coordinate ζ.

The total elastic energy functional to be minimized is then

$$\int_S h\langle U\rangle + \mathscr{P} \rightarrow \min \tag{4}$$

where $\mathscr{P}$ is the potential energy due to applied external forces and $\langle\bullet\rangle$ denotes the averaged integral through the thickness. Both, strain and potential, energies are needed to derive an asymptotically correct theory, and we will work with the total potential energy without splitting it into the strain and applied-load parts. Thus, the functional to be minimized can be represented as

$$\int_S J - \int_{\partial S} \langle P_\zeta^i u_i\rangle \rightarrow \min \tag{5}$$

where

$$J = h\langle U\rangle - h\langle F^i(x_j)u_i(x_\alpha, \zeta, h)\rangle - P_+^i(x_\alpha)u_i^+(x_\alpha, h) - P_-^i(x_\alpha)u_i^-(x_\alpha, h). \tag{6}$$

Here $P_\pm$, F and P_ζ are the applied external forces; S is the area occupied by plate in the $\{x_1, x_2\}$-plane and ∂S is the boundary of this area. The notations

$$\langle\bullet\rangle \equiv \int_{-1/2}^{+1/2} \bullet \,\mathrm{d}\zeta \quad \text{and} \quad \bullet^\pm \equiv \bullet_\pm \equiv \bullet|_{\zeta=\pm 1/2}$$

are introduced here.

The applied-load term written explicitly in eqn (5) accounts for applied loads on the edges of the plate. In this paper, we will not consider the edge-zone problem. This is an important subject in its own right and will be treated in a later paper. Therefore, we will drop the $\int_{\partial S}$ terms everywhere. As a first approximation for the actual plate boundary conditions the ones developed below can be used.

Also, the present development is not applicable if the displacement is prescribed on either the upper or lower surface of the plate. One must derive another theory in this case.

3. DEFINITION OF AN ASYMPTOTICALLY CORRECT THEORY

3.1. *Small parameters*

Asymptotical methods rest on the existence of small parameters. In a linear theory one supposes the strain to be small. In addition to this, we consider the thickness of the plate h to be smaller than the wavelength of any deformation of the plate. Denoting this wavelength by l, this means that $\bullet_{,\alpha} = (1/l)O(\bullet)$ and that h/l is considered to be small.

3.2. *Definitions*

To be complete, any plate theory must consist of three major parts that can be defined as follows:

(i) The main exchange rules: a definition of the 2-D variables in terms of 3-D ones, along with a set of relations that represent the form of the 3-D variables in terms of 2-D ones, representing a sort of one-to-one correspondence between 2-D and 3-D variables. As examples of main exchange rules, see eqns (9)–(11), eqns (38)–(40), or eqns (59)–(64).

(ii) The 2-D (or plate) energy: the total potential energy as a function only of 2-D plate variables from which the complete 2-D theory can be derived.

(iii) The recovering relations: approximations of the original 3-D variables (functions to be found) in terms of the 2-D plate variables.

Recovering relations always enable one to express the 3-D variables as a series with respect to h/l. However, those series are, in general, not explicit expansions of 3-D variables with respect to h/l. This is because the 2-D quantities that appear explicitly in the recovering relations may also depend on h/l, since the 2-D plate energy may depend on h/l. We will use term the final expansion to denote the result obtained after first expanding all quantities pertaining to the 2-D theory and then substituting them into the recovering relations.

The final expansion does not necessarily coincide with the expansion of 3-D variables obtained from an exact 3-D solution of any problem. This drives us to the definition of an asymptotically correct theory.

Definition: an asymptotically correct plate theory of a given order is one for which the asymptotic expansion of the exact solution coincides up to the given order with the final expansion.

The following remarks must be mentioned here.

—There is no unique plate theory of a given order.
—The form of the recovering relations and 2-D equations depends on how the 2-D variables are chosen.
—There is no universal rule on how to extract 2-D variables.

We also use the terminology that two different plate theories are said to be equivalent if each of them can be transferred to the other by a change of 2-D variables, and two plate theories are asymptotically equivalent if they have final expansions which are asymptotically correct up to the same order.

It is clear that the final expansions for equivalent theories are coincident, but two asymptotically equivalent plate theories may have different final expansions in higher-order terms that are not asymptotically correct.

We will discuss below how to develop plate theories that are asymptotically equivalent and correct up to the second order.

4. THE NECESSARY CONDITION FOR ASYMPTOTICAL CORRECTNESS

Supposing the recovering relations are known, we can write them symbolically:

$$u(x_\alpha, \zeta, h) = \mathscr{R}[v(x_\alpha, h), \zeta, h] \tag{7}$$

where v is the set of all plate variables and $\mathscr{R}$ is a given functional.

In order to derive the 2-D energy, one has to substitute eqn (7) into eqn (6), making use of eqns (2) and (3), and drop higher-order terms. The resulting 2-D energy density

$$\mathscr{E} = \text{main part of } J$$

will be a function of the 2-D strains, which we denote as ε.

To formulate the necessary conditions for a plate theory to be asymptotically correct let us represent the exact 3-D displacement in the form

$$u(x_\alpha, \zeta, h) = \mathscr{R}[v(x_\alpha), \zeta, h] + w(x_\alpha, \zeta, h) \tag{8}$$

where the warping w is the difference between the recovering relations and exact values of displacements.

Substituting eqn (8) into eqn (6), one obtains an expression that depends on the 2-D strains, ε, their derivatives, $\varepsilon_{,\alpha}$. Thus, the total potential can be split into four parts:

(i) the first part depends on 2-D strains, ε, only. This main part is the 2-D energy;
(ii) the second part consists of the leading interaction terms between ε, $\varepsilon_{,\alpha}$ and w, $w_{;\zeta}$ $w_{,\alpha}$;
(iii) the third part consists of the terms with $\varepsilon_{,\alpha}$ and the interaction terms between ε and $\varepsilon_{,\alpha}$;
(iv) the fourth part contains only higher-order terms and can be dropped.

The necessary condition for the theory to be asymptotically correct is that the second part of the total potential energy is equal to zero for any admissible warping and 2-D variables.

All of this is equivalent to the variational-asymptotical method, presented by Berdichevsky (1979) and successfully applied in a number of publications (see Berdichevsky and

Starosel'skii (1983) ; Le (1986)). Although it is not the usual way to present the variational-asymptotical method, we found the present way to be more convenient for the purposes of present work.

5. CLASSICAL THEORY

The most trivial plate theory of zero order can be constructed with the following choice of the main exchange rules

$$v_i(x_\alpha) = \langle u_i(x_\alpha, \zeta)\rangle \tag{9}$$

$$u_i(x_\alpha, \zeta, h) = v_i(x_\alpha, h) + w_i(x_\alpha, \zeta, h) \tag{10}$$

$$\langle w_i(x_\alpha, \zeta, h)\rangle = 0. \tag{11}$$

The recovering relations are (see Berdichevsky (1979))

$$\begin{aligned} u_\alpha &= v_\alpha - h\zeta v_{3,\alpha} \\ u_3 &= v_3 + hD_e^{\gamma\delta}(\zeta)A_{\gamma\delta} + hD_b^{\gamma\delta}(\zeta)B_{\gamma\delta} \end{aligned} \tag{12}$$

where A and B are 2-D in-plane and bending measure of plate deformation

$$A_{\gamma\delta} = \tfrac{1}{2}(v_{\gamma,\delta} + v_{\delta,\gamma}) \tag{13}$$

$$B_{\gamma\delta} = -hv_{3,\gamma\delta}. \tag{14}$$

Tensors $D_e^{\gamma\delta}(\zeta)$ and $D_b^{\gamma\delta}(\zeta)$ are the integrals of the following equations

$$\begin{aligned} D_{e,\zeta}^{\gamma\delta} &= -C^{\gamma\delta}\langle D_e^{\gamma\delta}\rangle = 0 \\ D_{b,\zeta}^{\gamma\delta} &= -\zeta C^{\gamma\delta}\langle D_b^{\gamma\delta}\rangle = 0. \end{aligned} \tag{15}$$

The plate 2-D total energy is

$$\mathscr{E}_{cl} = E_{cl} + P_{cl}$$

where strain, E_{cl}, and potential, P_{cl}, energies are

$$\begin{aligned} E_{cl} &= \tfrac{1}{2}hE_e^{\alpha\beta\gamma\delta}A_{\alpha\beta}A_{\gamma\delta} + \tfrac{1}{2}hE_b^{\alpha\beta\gamma\delta}B_{\alpha\beta}B_{\gamma\delta} + hE_{eb}^{\alpha\beta\gamma\delta}A_{\alpha\beta}B_{\gamma\delta} \\ P_{cl} &= -f^i v_i + m^\alpha v_{3,\alpha} \end{aligned} \tag{16}$$

where

$$\begin{aligned} E_e^{\alpha\beta\gamma\delta} &= \langle D^{\alpha\beta\gamma\delta}\rangle \\ E_{eb}^{\alpha\beta\gamma\delta} &= \langle \zeta D^{\alpha\beta\gamma\delta}\rangle \\ E_b^{\alpha\beta\gamma\delta} &= \langle \zeta^2 D^{\alpha\beta\gamma\delta}\rangle \\ f^i &= P_+^i + P_-^i + h\langle F^i\rangle \\ m^\alpha &= \tfrac{1}{2}h(P_+^\alpha - P_-^\alpha) + h^2\langle \zeta F^\alpha\rangle. \end{aligned} \tag{17}$$

The full functional to be minimized is

$$\int_S (E_{el}+P_{el}) + \int_{\partial S} (-f_\zeta^i v_i + m_\zeta^\alpha v_{3,\alpha}) \tag{18}$$

where

$$f_\zeta^i = h\langle P_\zeta^i \rangle \quad m_\zeta^\alpha = h^2 \langle \zeta P_\zeta^\alpha \rangle. \tag{19}$$

In order to verify that the presented theory is asymptotically correct up to the order $(h/l)^0$, let us write the true 3-D displacements in the form†

$$\begin{aligned} u_\alpha &= \underset{l\varepsilon}{v_\alpha} - \underset{h\varepsilon}{h\zeta v_{3,\alpha}} + \underset{l(h/l)^2\varepsilon}{w_\alpha} \\ u_3 &= \underset{h(h/l)^{-2}\varepsilon}{v_3} + \underset{h\varepsilon}{hD_e : A} + \underset{h\varepsilon}{hD_b : B} + \underset{h(h/l)\varepsilon}{w_3} \end{aligned} \tag{20}$$

where w is the warping, satisfied by eqn (11). The asymptotical order of each term is written under it here.

The strains, calculated from eqn (20), are‡

$$\varepsilon_{\alpha\beta} = \underset{\varepsilon}{A_{\alpha\beta}} + \underset{\varepsilon}{\zeta B_{\alpha\beta}} + \underset{(h/l)^2\varepsilon}{w_{\underline{\alpha},\underline{\beta}}} \tag{21}$$

$$2\varepsilon_{\alpha 3} = \underset{(h/l)\varepsilon}{\frac{1}{h} w_{\alpha,\zeta}} + \underset{(h/l)\varepsilon}{hD_e : A_{,\alpha}} + \underset{(h/l)\varepsilon}{hD_b : K_{,\alpha}} + \underset{(h/l)^2\varepsilon}{w_{3,\alpha}} \tag{22}$$

$$\varepsilon_{33} = \underset{(h/l)\varepsilon}{\frac{1}{h} w_{3,\zeta}} + \underset{\varepsilon}{D_{e,\zeta} : A} + \underset{\varepsilon}{D_{b,\zeta} : B} \tag{23}$$

Note that only the terms of order $(h/l)^0\varepsilon$ are asymptotically correct.

After substituting eqn (20) into the 3-D energy density eqn (3) one gets

$$\begin{aligned} U = {} & D^{\alpha\beta\gamma\delta}(A_{\alpha\beta}+\zeta B_{\alpha\beta})(A_{\gamma\delta}+\zeta B_{\gamma\delta}) \\ & + \tfrac{1}{2} D_\perp [\underline{\underline{(D_{e,\zeta}^{\gamma\delta} + C^{\gamma\delta})}} A_{\gamma\delta} + \underline{\underline{(D_{b,\zeta}^{\gamma\delta} + \zeta C^{\gamma\delta})}} B_{\gamma\delta} \\ & + \frac{1}{h} w_{3,\zeta} + hC^{\gamma\delta} D_{e_{\underline{\zeta}}} : A_{,\underline{\delta}} + hC^{\gamma\delta} D_{b_{\underline{\zeta}}} : B_{,\underline{\delta}} + C^{\gamma\delta} w_{\underline{\gamma},\underline{\delta}}]^2 \\ & + D_\perp^{\alpha\beta}\left(\frac{1}{h} w_{\alpha,\zeta} + hD_e : A_{,\alpha} + hD_b : B_{,\alpha} + w_{3,\alpha}\right)(\alpha \to \beta) \\ & + D^{\alpha\beta\gamma\delta}(A_{\alpha\beta}+\zeta B_{\alpha\beta}) w_{\underline{\gamma},\underline{\delta}} + \tfrac{1}{2} D^{\alpha\beta\gamma\delta} w_{\underline{\alpha},\underline{\beta}} w_{\underline{\gamma},\underline{\delta}}. \end{aligned} \tag{24}$$

The double underlined terms are equal to zero due to eqns (15). They are the only terms that produce part 2 of the energy which needs to be made equal to zero.

The 2-D equilibrium equations are

† The notations : or · that denote contracting with respect to two or one indices is introduced here ($D_b : B \equiv D_b^{\gamma\delta} B_{\gamma\delta}$ for example). We also will sometimes drop the indices of a tensor if an expression is clear without them (for example, $D^{\alpha\beta\gamma\delta}$, $D^{\alpha\beta}$ and D denote the same objects).

‡ The underlining of indices indicates the operation of symmetrization with respect to underlined indices. For example, $w_{\underline{\alpha},\underline{\beta}} \equiv \frac{1}{2}(w_{\alpha,\beta} + w_{\beta,\alpha})$.

$$-N^{\alpha\beta}_{,\beta} = f^{\alpha} \quad M^{\alpha\beta}_{,\alpha\beta} = f^{3} + m^{\alpha}_{,\alpha} \tag{25}$$

where

$$N^{\alpha\beta} = \frac{\partial E_{cl}}{\partial A_{\alpha\beta}} \quad M^{\alpha\beta} = h\frac{\partial E_{cl}}{\partial B_{\alpha\beta}}. \tag{26}$$

The choice of variables, eqn (9), with the main exchange rule, eqn (10), is not unique. Plate displacement might be the displacement of the middle plane, for example, or it can be the displacement of either the upper or lower surface.

With any choice of 2-D variables, any theory that keeps the asymptotically leading terms asymptotically correct will always be asymptotically equivalent to the theory described above.

6. ORDER OF QUANTITIES

Since the problem under consideration is linear, the orders of all quantities can be synchronously increased or decreased, and we have to make some agreement of how to assign the orders. We will assume the order of plate deformations A and B to be ε and order of any quantity can be expressed as $(h/l)^n\varepsilon$. We have already used this notation in eqns (20–23).

Material coefficients are treated as independent of h quantities. We denote their characteristic value by μ to facilitate understanding of some expressions, but no expansion with respect to μ is considered.

From the classical equations it can be also derived that

$$f^{3} \sim \mu\left(\frac{h}{l}\right)^{2}\varepsilon \quad f^{\alpha} \sim \mu\left(\frac{h}{l}\right)\varepsilon \quad m^{\alpha} \sim \mu h\left(\frac{h}{l}\right)\varepsilon \tag{27}$$

and

$$P^{3}_{\pm} \sim hF^{3} \sim \mu\left(\frac{h}{l}\right)^{2}\varepsilon \quad P^{\alpha}_{\pm} \sim hF^{\alpha} \sim \mu\left(\frac{h}{l}\right)\varepsilon. \tag{28}$$

We will construct a Reissner-like theory to produce quantities of order $(h/l)^2\varepsilon$ that are asymptotically correct.

7. REISSNER EQUATIONS

Reissner theory (see Reissner (1944, 1945)) is based on five 2-D variables to be found: three displacements, v_i, and two rotations, θ_α. The plate energy depends on three kinds of strain:

$$A_{\alpha\beta} = v_{\underline{\alpha,\beta}}\text{—in-plane deformation} \tag{29}$$

$$K_{\alpha\beta} = h\theta_{\underline{\alpha,\beta}}\text{—bending deformation} \tag{30}$$

$$\gamma_\alpha = \theta_\alpha + v_{3,\alpha}\text{—transverse shear deformation.} \tag{31}$$

The 2-D equilibrium equations are

$$-N^{\alpha\beta}_{,\beta} = f^{\alpha} \tag{32}$$

$$-M^{\alpha\beta}_{,\beta} + Q^{\alpha} = m^{\alpha} \tag{33}$$

$$-Q^{\alpha}_{;\alpha} = f^{3} \tag{34}$$

where

$$N^{\alpha\beta} = \frac{\partial \mathscr{E}}{\partial A_{\alpha\beta}} \quad M^{\alpha\beta} = h\frac{\partial \mathscr{E}}{\partial K_{\alpha\beta}} \quad Q^{\alpha} = \frac{\partial \mathscr{E}}{\partial \gamma_{\alpha}} \tag{35}$$

and where

$$\mathscr{E} = E_R + P_R$$

with the strain energy density being

$$E_R = \tfrac{1}{2}hE_e^{\alpha\beta;\gamma\delta}A_{\alpha\beta}A_{\gamma\delta} + \tfrac{1}{2}hE_b^{\alpha\beta;\gamma\delta}K_{\alpha\beta}K_{\gamma\delta} + hE_{eb}^{\alpha\beta;\gamma\delta}A_{\alpha\beta}K_{\gamma\delta} + \tfrac{1}{2}hG^{\alpha\beta}\gamma_\alpha\gamma_\beta \tag{36}$$

and the potential energy density of the applied loads being

$$P_R = -f^i v_i - m^\alpha \theta_\alpha. \tag{37}$$

Here† $E_{(e;b)}$, f^i and m^α are given by eqn (17), and G is the tensor of transverse shear stiffnesses.

The Reissner equations are very attractive, since each term has a clear physical meaning, and it is convenient to base finite element methods on them. Unfortunately, Reissner's original recovering relations are written in terms of stresses only. Thus, it is difficult to judge their asymptotical correctness. To do so in this present framework requires that the Reissner equations be complemented with appropriate recovering relations. The first successful attempt for homogeneous plates was made by Berdichevsky (1979). He started from the following main exchange rules

$$v_i = \langle u_i \rangle, \quad \theta_\alpha = \frac{1}{h}\langle 12\zeta u_\alpha \rangle \tag{38}$$

$$u_\alpha = v_\alpha + h\zeta\theta_\alpha + w_\alpha, \quad u_3 = v_3 + w_3 \tag{39}$$

$$\langle w_i \rangle = 0, \quad \langle \zeta w_\alpha \rangle = 0 \tag{40}$$

and proved that the 2-D total energy

$$\mathscr{E}_R = \tfrac{1}{2}hE_e^{\alpha\beta;\gamma\delta}A_{\alpha\beta}A_{\gamma\delta} + \tfrac{1}{2}hE_b^{\alpha\beta;\gamma\delta}K_{\alpha\beta}K_{\gamma\delta} + hE_{eb}^{\alpha\beta;\gamma\delta}A_{\alpha\beta}K_{\gamma\delta} + \tfrac{1}{2}hG^{\alpha\beta}\gamma_\alpha\gamma_\beta + P_e : A + P_b : K - f^i v_i - m^\alpha\theta_\alpha \tag{41}$$

where

$$P_e = \tfrac{1}{2}h(P_3^+ - P_3^-)C \tag{42}$$

$$P_b = \tfrac{1}{10}h(P_3^+ + P_3^-)C \tag{43}$$

with the recovering relations

$$u_\alpha = v_\alpha - h\zeta v_{3,\alpha} - 5h\left(\frac{\zeta^3}{3} - \frac{\zeta}{4}\right)\gamma_\alpha + h^2\tfrac{1}{2}(\zeta^2 - \tfrac{1}{12})C : A_{,\alpha} + h^2\frac{1}{2}\left(\frac{\zeta^3}{3} - \frac{\zeta}{20}\right)C : K_{,\alpha} \tag{44}$$

$$u_3 = v_3 - h\zeta C : A - h\tfrac{1}{2}(\zeta^2 - \tfrac{1}{20})C : K \tag{45}$$

† Notation $E_{(e;b)}$ is read as either E_e or E_b, or E_{eb}.

gives a theory which is asymptotically correct up to the second order. Here $P_e : A + P_b : K$ is the contribution to the energy from external forces.†

The above theory was derived under the assumption of upper and lower surface in-plane tractions and body forces set equal to zero. This result can be extended to a special case of inhomogeneity, namely, if all the material coefficients are proportional to the same function of through-the-thickness coordinate, ζ. Although the latter case is impractical, it can be used as a test problem for debugging purposes. Another special case is pointed by Atilgan and Hodges (1992). No other asymptotically correct theories for laminated plates are known to the authors.

Taking into account the beauty of the Reissner equations, we will try to find the recovering relations for them in the generic case that are as close to asymptotical correctness as possible. For the purpose of our discussion, let us call any 2-D plate theory containing eqns (29)–(35) and (41) a *Reissner-like theory*.

8. REISSNER-LIKE THEORY FOR INHOMOGENEOUS PLATES WITH ORTHOTROPIC LAMINAE

In this section we present a Reissner-like theory for laminated plates, each lamina of which is made of an orthotropic material. All the expressions given here will be derived in the next section.

The strain and potential energies are represented by eqns (36) and (37) with $E_{(e;b)}$, f and m given by eqn (17). Quantities $P_{(e;b)}$ are‡

$$P_{(e;b)} = -P^3_{\pm} hD^{\pm}_{(e;b)} - h^2\langle F^3 D_{(e;b)}\rangle + (h^2 P^\alpha_{\pm} H^{\nu\pm}_{(e;b)\alpha} + h^3\langle F^\alpha H^\nu_{(e;b)\alpha}\rangle)_{,\nu} \tag{46}$$

with recovering relations being

$$\begin{aligned} u_\alpha &= v_\alpha - h\zeta v_{3,\alpha} + h^2 H^\nu_{e\alpha} : A_{,\nu} + h^2 H^\nu_{b\alpha} : K_{,\nu} + hR_\alpha \\ u_3 &= v_3 + hD_e : A + hD_b : K - h^2 D^{\alpha\nu}_b \gamma_{\alpha,\nu} + h^3 J^{\mu\nu}_e : A_{,\mu\nu} + h^3 J^{\mu\nu}_b : K_{,\mu\nu} + hR_3 \end{aligned} \tag{47}$$

where

$$\begin{aligned} D_{(e;b)} &= \bar{D}_{(e;b)}(\zeta) + D^0_{(e;b)} \\ H^\nu_{(e;b)\alpha} &= \bar{H}^\nu_{(e;b)\alpha}(\zeta) + H^{0\nu}_{(e;b)\alpha} - \zeta\delta^\nu_\alpha D^0_{(e;b)} \end{aligned} \tag{48}$$

and where $\bar{D}_{(e;b)}(\zeta)$, $\bar{H}^\nu_{(e;b)\alpha}(\zeta)$, $J_{(e;b)}(\zeta)$, $R_\alpha(\zeta)$ and $R_3(\zeta)$ are the solutions of the equations

$$\bar{D}^{\gamma\delta}_{e,\zeta} = -C^{\gamma\delta}, \quad \bar{D}^{\gamma\delta}_{b,\zeta} = -\zeta C^{\gamma\delta}, \quad \langle \bar{D}^{\gamma\delta}_{(e;b)}\rangle = 0 \tag{49}$$

$$\bar{H}^\nu_{(e;b)\alpha,\zeta} = \bar{L}^\nu_{(e;b)\alpha} - \delta^\nu_\alpha \bar{D}_{(e;b)}, \quad \langle \bar{H}^\nu_{(e;b)\alpha}\rangle = 0 \tag{50}$$

$$\begin{aligned} [D_\perp(J^{\mu\nu}_{(e;b),\zeta} + C^{\alpha\mu} H^\nu_{(e;b)\alpha})]_{,\zeta} + D^{\alpha\mu}_\angle \bar{L}^\nu_{(e;b)\alpha} &= \langle D^{\alpha\mu}_\angle \bar{L}^\nu_{(e;b)\alpha}\rangle, \quad \langle J^{\mu\nu}_{(e;b)}\rangle = 0 \\ [D_\perp(J^{\mu\nu}_{(e;b),\zeta} + C^{\alpha\mu} H^\nu_{(e;b)\alpha})]\Big|_{\zeta=\pm 1/2} &= 0 \end{aligned} \tag{51}$$

$$(D^{\alpha\beta}_\angle R_{\beta,\zeta})_{,\zeta} + hF^\alpha = f^\alpha, \quad \langle R_\alpha\rangle = 0, \quad (D^{\alpha\beta}_\angle R_{\beta,\zeta})\Big|_{\zeta=\pm 1/2} = \pm P^\alpha_\pm \tag{52}$$

$$[D_\perp(R_{3,\zeta} + C^{\alpha\beta} R_{\alpha,\beta})]_{,\zeta} + D^{\alpha\beta}_\angle R_{\alpha,\beta} + hF^3 = f^3 + \langle D^{\alpha\beta}_\angle R_{\alpha,\beta}\rangle, \quad \langle R_3\rangle = 0$$

† Actually some parts of this contribution come from the 3-D strain energy and some from the 3-D potential energy. That is why we work with the total potential energy without splitting it into the strain and applied-load parts.

‡ Notation $P_\pm(\bullet)^\pm$ means $P_+(\bullet)^+ + P_-(\bullet)^-$.

$$[D_{\perp}(R_{3,\zeta}+C^{\alpha\beta}R_{\alpha,\beta})]\Big|_{\zeta=\pm 1/2} = \pm P^{3}_{\pm} \tag{53}$$

and finally where $\bar{L}^{\gamma}_{(e;b)\alpha}$ are the solution of

$$(D^{\alpha\mu}_{\angle}\bar{L}^{\beta;\delta}_{e\mu})_{,\zeta}+D^{\alpha\beta;\delta} = \langle D^{\alpha\beta;\delta}\rangle, \quad (D^{\alpha\mu}_{\angle}\bar{L}^{\beta;\delta}_{e\mu})\Big|_{\zeta=\pm 1/2} = 0 \tag{54}$$

$$(D^{\alpha\mu}_{\angle}\bar{L}^{\beta;\delta}_{b\mu})_{,\zeta}+\zeta D^{\alpha\beta;\delta} = \langle \zeta D^{\alpha\beta;\delta}\rangle, \quad (D^{\alpha\mu}_{\angle}\bar{L}^{\beta;\delta}_{b\mu}) = 0\Big|_{\zeta=\pm 1/2} = 0. \tag{55}$$

Equations (49)–(55) make part 2 of the energy equal to zero and provide asymptotical correctness. The following expression, part 3 of the energy, must be zero in order for the theory to be Reissner-like:

$$\begin{aligned} E_3 = {} & \tfrac{1}{2}G^{-1}_{\alpha\beta}(E^{\alpha\gamma}_{eb}:A_{,\gamma}+E^{\alpha\gamma}_{b}:K_{,\gamma})(\alpha\to\beta) \\ & +\tfrac{1}{2}\langle D^{\alpha\beta}_{\angle}(\zeta)[\bar{L}^{\gamma}_{e\alpha}(\zeta):A_{,\gamma}+\bar{L}^{\gamma}_{b\alpha}(\zeta):K_{,\gamma}][\alpha\to\beta]\rangle \\ & -\langle D^{\alpha\beta}(\zeta):(A_{,\beta}+\zeta K_{,\beta})[\bar{H}^{\gamma}_{e\alpha}(\zeta):A_{,\gamma}+\bar{H}^{\gamma}_{b\alpha}(\zeta):K_{,\gamma} \\ & +(H^{0\gamma}_{e\alpha}-\zeta\delta^{\gamma}_{\alpha}D^{0}_{e}):A_{,\gamma}+(H^{0\gamma}_{b\alpha}-\zeta\delta^{\gamma}_{\alpha}D^{0}_{b}):K_{,\gamma}]\rangle. \end{aligned} \tag{56}$$

This is a 12×12 quadratic form with respect to $A_{,\gamma}$ and $K_{,\gamma}$ depending on 33 variables, $D^{0}_{(e;b)}$, $H^{0}_{(e;b)}$ and G, if all the above equations are solved. This form contains 78 coefficients, and in the general case it is not possible to make E_3 zero. However, real plates frequently have some symmetry that can help to drive E_3 to zero. At least one can numerically minimize the sum of the squares of the 78 coefficients by choosing 33 variables. We will discuss later some of the test calculations we have done so far.

8.1. *Computational notes*

In order to solve the system of eqns (49)–(55) one should basically have two solvers. One of them has to solve the equation for $f(\zeta)$

$$f(\zeta)_{,\zeta} = g(\zeta)-\langle g(\zeta)\rangle \quad f\Big|_{\zeta=\pm 1/2} = 0 \tag{57}$$

with arbitrary $g(\zeta)$, and another has to give the solution of the equation for $f(\zeta)$

$$f(\zeta)_{,\zeta} = g(\zeta) \quad \langle f(\zeta)\rangle = 0. \tag{58}$$

Using solver eqn (57), one can start with eqns (54), and (55), then solve eqns (49) and (50) via eqn (58). Then the quadratic form eqn (56) must be minimized to get values of $D^{0}_{(e;b)}$, $H^{0}_{(e;b)}$ and G. After that one is ready to solve eqn (51) using the combination of the solvers in eqns (57) and (58). Equation (52)–(53) can also be solved by the combination of solvers in eqns (57) and (58), but they depend on external forces and are, therefore, part of the 2-D problem.

This procedure is not computationally expensive.

9. DERIVATION

In order to keep the composition of eqns (29–35), the main exchange rules should be of the form

$$v_i = \langle u_i \rangle \quad \theta_\alpha = \frac{1}{h}\langle \psi^\beta_\alpha u_\beta \rangle \tag{59}$$

$$u_\alpha = v_\alpha - h\zeta v_{3,\alpha} + h\phi^\nu_\alpha(\zeta)\gamma_\nu + w_\alpha \quad u_3 = v_3 + w_3 \tag{60}$$

$$\langle w_i \rangle = 0 \tag{61}$$

$$\langle \psi^\beta_\alpha w_\beta \rangle = 0 \tag{62}$$

where $\psi(\zeta)$ and $\phi(\zeta)$ can be arbitrary within the restrictions

$$\langle \psi^\beta_\alpha \rangle = 0 \quad \langle \phi^\beta_\alpha \rangle = 0 \tag{63}$$

$$\langle \psi^\nu_\alpha \phi^\beta_\nu \rangle = \langle \zeta \psi^\beta_\alpha \rangle = \delta^\beta_\alpha . \tag{64}$$

The specific form of eqns (63) and (64) is not necessary, but this is a convenient way to normalize θ_α and separate them from $v_{3,\alpha}$. Any other relation would lead to an equivalent theory.

In order to simplify all the expressions, let us introduce the object A^a as the pair of A and K. The quasi index a can be either e or b. If quasi index a equals e then

$$A^e \equiv A$$

and if quasi index a equals b then

$$A^b \equiv K.$$

We will use the summation rule with respect to this quasi index. For example,

$$J_a : A^a \equiv J_e : A + J_b : K.$$

Also eqn (36) can be written in this notation as

$$E_R = \frac{h}{2} A^a : E_{ac} : A^c + \frac{h}{2}\gamma \cdot G \cdot \gamma + P_a : A^a .$$

We assumed here that $E_{ee} \equiv E_e$ and $E_{bb} \equiv E_b$.

We will also need the special notation $\mathscr{I}_c$ defined as

$$\mathscr{I}_e \equiv 1 \quad \mathscr{I}_b \equiv \zeta .$$

Let the recovering relation be of the following general form

$$u_\alpha = \underbrace{v_\alpha}_{h(h/l)^{-1}\varepsilon} - \underbrace{h\zeta v_{3,\alpha}}_{h(h/l)^{-1}\varepsilon} + \underbrace{h\phi_\alpha \cdot \gamma}_{h(h/l)\varepsilon} + \underbrace{h\tilde{R}_\alpha}_{h(h/l)\varepsilon} + \underbrace{h^2 \tilde{H}^\nu_{c\alpha} : A^c_\nu}_{h(h/l)\varepsilon} + \underbrace{w_\alpha}_{h(h/l)^3\varepsilon} \tag{65}$$

$$u_3 = \underbrace{v_3}_{h(h/l)^{-2}\varepsilon} + \underbrace{hD_c : A^c}_{h\varepsilon} + \underbrace{h\tilde{R}_3}_{h(h/l)^2\varepsilon} + \underbrace{h^2 \tilde{\Gamma}^{\alpha\beta}\gamma_{\alpha,\beta}}_{h(h/l)^2\varepsilon} + \underbrace{h^3 \tilde{J}^{\mu\nu}_c : A^c_{,\mu\nu}}_{h(h/l)^2\varepsilon} + \underbrace{w_3}_{h(h/l)^4\varepsilon} \tag{66}$$

where $\tilde{R}_i(x_\alpha, \zeta)$ are terms coming from external forces, $D_c(\zeta)$, $\tilde{H}_c(\zeta)$, $\tilde{\Gamma}(\zeta)$ and $J_c(\zeta)$ are functions of ζ to be determined. Here, we put all the possible terms that are needed to get order $(h/l)^2\varepsilon^2$ in the energy. The orthotropic symmetry of the laminae is taken into account here. It explains the absence of terms with $\gamma_{,\nu}$, A^a and $A^a_{,\mu\nu}$ in eqn (65), and the absence of terms with $A^a_{,\mu}$ and γ in eqn (66).

Strains are defined according to relations (65, 66)

$$\varepsilon_{\alpha\beta} = \underset{\varepsilon}{A_{\alpha\beta}} + \underset{\varepsilon}{\zeta K_{\alpha\beta}} - \underset{(h/l)^2\varepsilon}{h\zeta\gamma_{\underline{\alpha,\beta}}} + \underset{(h/l)^2\varepsilon}{h\phi_{\underline{\alpha}}\cdot\gamma_{,\underline{\beta}}} + \underset{(h/l)^2\varepsilon}{h\tilde{R}_{\underline{\alpha,\beta}}} + \underset{(h/l)^2\varepsilon}{h^2\tilde{H}^{v}_{c\alpha} : A^{c}_{,v\underline{\beta}}} + \underset{(h/l)^4\varepsilon}{w_{\underline{\alpha,\beta}}} \tag{67}$$

$$2\varepsilon_{\alpha 3} = \underset{(h/l)^3\varepsilon}{\frac{1}{h}w_{\alpha,\zeta}} + \underset{(h/l)\varepsilon}{\phi_{\alpha,\zeta}\cdot\gamma} + \underset{(h/l)\varepsilon}{\tilde{R}_{\alpha,\zeta}} + \underset{(h/l)\varepsilon}{h\tilde{L}^{v}_{c\alpha} : A^{c}_{,v}} + \underset{(h/l)^3\varepsilon}{h\tilde{R}_{3,\alpha}} + \underset{(h/l)^3\varepsilon}{h^2\tilde{\Gamma}^{\mu v}\gamma_{\mu,v\alpha}} + \underset{(h/l)^3\varepsilon}{h^3\tilde{J}^{\mu v}_{c} : A^{c}_{,\mu v\alpha}} + \underset{(h/l)^3\varepsilon}{w_{3,\alpha}} \tag{68}$$

$$\varepsilon_{33} = \underset{(h/l)^4\varepsilon}{\frac{1}{h}w_{3,\zeta}} + \underset{\varepsilon}{D_{c,\zeta} : A^{c}} + \underset{(h/l)^2\varepsilon}{\tilde{R}_{3,\zeta}} + \underset{(h/l)^2\varepsilon}{h\tilde{\Gamma}^{\alpha\beta}_{,\zeta}\gamma_{\alpha,\beta}} + \underset{(h/l)^2\varepsilon}{h^2\tilde{J}^{\mu v}_{c,\zeta} : A^{c}_{,\mu v}} \tag{69}$$

where notation

$$\tilde{L}^{v}_{c\alpha} = \tilde{H}^{v}_{c\alpha,\zeta} + D_c\delta^{v}_{\alpha} \tag{70}$$

is introduced. The asymptotical order of each term is written under it here. Relation $-hw_{3,\alpha\beta} \equiv K_{\alpha\beta} - h\gamma_{\alpha,\beta}$ has also been used in eqn (67). Only terms up to the order $(h/l)^2\varepsilon$ are expected to be asymptotically correct.

Now, one can substitute eqns (66)–(69) into the 3-D total energy density, eqns (6), and get

$$\begin{aligned}
J = {} & E_{cl} + P_R + m^{\alpha}\gamma_{\alpha} + h\langle \tfrac{1}{2}D_{-}[(D_{c,\zeta} + \mathscr{I}_c C) : A^c]^2 + D_{\perp}[(D_{c,\zeta} + \mathscr{I}_c C) : A^c] \\
& \times [h^2 J^{\mu v}_{c,\zeta} : A^{c}_{,\mu v} + \tilde{R}_{3,\zeta} + h\tilde{\Gamma}^{\alpha\beta}_{,\zeta}\gamma_{\alpha,\beta} + C^{\alpha\beta}w_{\underline{\alpha,\beta}} \\
& + C^{\alpha\beta}(h\phi_{\underline{\beta}}\cdot\gamma_{,\underline{\alpha}} - h\zeta\gamma_{\underline{\alpha,\beta}} + h\tilde{R}_{\underline{\alpha,\beta}} + h^2\tilde{H}^{v}_{c\alpha} : A^{c}_{,v\beta})] + D_{\perp}\frac{1}{h}w_{3,\zeta}[(D_{c,\zeta} + \mathscr{I}_c C) : A^c] \\
& + D_{\perp}\frac{1}{h}w_{3,\zeta}[h^2\tilde{J}^{\mu v}_{c,\zeta} : A^{c}_{,\mu v} + \tilde{R}_{3,\zeta} + h\tilde{\Gamma}^{\alpha\beta}_{,\zeta}\gamma_{\alpha,\beta} \\
& + C^{\alpha\beta}(h\phi_{\beta}\cdot\gamma_{,\underline{\alpha}} - h\zeta\gamma_{\underline{\alpha,\beta}} + h\tilde{R}_{\underline{\alpha,\beta}} + h^2\tilde{H}^{v}_{c\alpha} : A^{c}_{,v\beta})] + \tfrac{1}{2}D^{\alpha\beta}_{\angle}(\phi_{\alpha,\zeta}\cdot\gamma)(\phi_{\beta,\zeta}\cdot\gamma) \\
& + D^{\alpha\beta}_{\angle}(\phi_{\alpha,\zeta}\cdot\gamma)(h\tilde{L}^{v}_{c\beta} : A^{c}_{,v} + \tilde{R}_{\beta,\zeta}) + \tfrac{1}{2}D^{\alpha\beta}_{\angle}(h\tilde{L}^{v}_{c\alpha} : A^{c}_{,v} + \tilde{R}_{\alpha,\zeta})(\alpha \to \beta) \\
& + D^{\alpha\beta}_{\angle}\frac{1}{h}w_{\beta,\zeta}(\phi_{\alpha,\zeta}\cdot\gamma + \tilde{R}_{\alpha,\zeta} + h\tilde{L}^{v}_{c\alpha} : A^{c}_{,v}) + D^{\alpha\beta}_{\angle}w_{3,\beta}(\phi_{\alpha,\zeta}\cdot\gamma + \tilde{R}_{\alpha,\zeta} + h\tilde{L}^{v}_{c\alpha} : A^{c}_{,v}) \\
& + D^{\alpha\beta;\gamma\delta}_{|}(A_{\gamma\delta} + \zeta K_{\gamma\delta}) \times (h\phi_{\alpha}\cdot\gamma_{,\beta} - h\zeta\gamma_{\alpha,\beta} + h\tilde{R}_{\alpha,\beta} + h^2\tilde{H}^{v}_{c\alpha} : A^{c}_{,v\beta} + w_{\alpha,\beta})\rangle \\
& - P^{\alpha}_{\pm}(h\phi_{\alpha}\cdot\gamma + h^2\tilde{H}^{v}_{c\alpha} : A^{c}_{,v} + w_{\alpha})^{\pm} - \langle hF^{\alpha}(h\phi_{\alpha}\cdot\gamma + h^2\tilde{H}^{v}_{c\alpha} : A^{c}_{,v} + w_{\alpha})\rangle \\
& - P^{3}_{\pm}(hD_c : A^c + w_3)^{\pm} - \langle hF^3(hD_c : A^c + w_3)\rangle.
\end{aligned} \tag{71}$$

Notation $P_{\pm}(\bullet)^{\pm}$ means $P_{+}(\bullet)^{+} + P_{-}(\bullet)^{-}$ here. All the higher-order terms from part 3 and 4 of energy are dropped here.

Let us start the investigation with the interaction term between w_3 and A^c:

$$\langle D_{\perp}w_{3,\zeta}[(D_{c,\zeta} + \mathscr{I}_c C) : A^c]\rangle. \tag{72}$$

This must be zero for any admissible function $w_3(\zeta)$ satisfied by eqn (61). One can derive that

$$[D_{\perp}((D_{c,\zeta} + \mathscr{I}_c C) : A^c)]_{,\zeta} = \lambda \tag{73}$$

$$[D_{\perp}((D_{c,\zeta} + \mathscr{I}_c C) : A^c)]|_{\zeta = \pm 1/2} = 0 \tag{74}$$

where λ is Lagrange multiplier that enforces the constraint $\langle w_3\rangle = 0$, eqn (61). In order to calculate λ one can integrate eqn (73) through the range of ζ and take into account eqn (74). It makes λ equal to zero. This procedure is simple, and we will skip it in the all similar derivations below. It means that the calculated value of Lagrange multipliers that enforce the constraints $\langle w_i \rangle = 0$, will always be substituted in the expressions below.

Since A^c is arbitrary,

$$D_{c,\zeta} + \mathscr{I}_c C = 0. \tag{75}$$

The solution of these equations can be written as

$$D_c(\zeta) = \bar{D}_c(\zeta) + D_c^0 \quad \langle \bar{D}_c \rangle = 0 \tag{76}$$

where $\bar{D}_c$ are the same as in classical theory (see eqn 14) and D_c^0 are arbitrary constants that are to be determined later.

The analogous derivation shows that the following equations will cause other terms of energy containing w_3 to vanish

$$[D_\perp(\tilde{J}^{\mu\nu}_{c,\zeta} + C^{\alpha\mu}\tilde{H}^{\nu}_{c\alpha})]_{,\zeta} + D^{\alpha\mu}_{\angle}\tilde{L}^{\nu}_{c\alpha} = \langle D^{\alpha\mu}_{\angle}\tilde{L}^{\nu}_{c\alpha}\rangle \tag{77}$$

$$[D_\perp(\tilde{J}^{\mu\nu}_{c,\zeta} + C^{\alpha\mu}\tilde{H}^{\nu}_{c\alpha})]|_{\zeta=\pm 1/2} = 0, \quad [D_\perp(\tilde{\Gamma}^{\mu\nu}_{,\zeta} + C^{\alpha\mu}\phi^{\nu}_{\alpha} - \zeta C^{\mu\nu})]_{,\zeta} + D^{\alpha\mu}_{\angle}\phi^{\nu}_{\alpha} = \langle D^{\alpha\mu}_{\angle}\phi^{\nu}_{\alpha}\rangle \tag{78}$$

$$\begin{aligned} &[D_\perp(\tilde{\Gamma}^{\mu\nu}_{,\zeta} + C^{\alpha\mu}\phi^{\nu}_{\alpha} - \zeta C^{\mu\nu})]|_{\zeta=\pm 1/2} = 0 \\ &[D_\perp(\tilde{R}_{3,\zeta} + C^{\alpha\beta}\tilde{R}_{\alpha,\beta})]_{,\zeta} + D^{\alpha\beta}_{\angle}\tilde{R}_{\alpha,\beta} + hF^3 = f^3 + \langle D^{\alpha\beta}_{\angle}\tilde{R}_{\alpha,\beta}\rangle \\ &[D_\perp(\tilde{R}_{3,\zeta} + C^{\alpha\beta}\tilde{R}_{\alpha,\beta})]|_{\zeta=\pm 1/2} = \pm P^3_{\pm}. \end{aligned} \tag{79}$$

Let us collect the terms containing w_α in eqn (71)

$$-P^{\alpha}_{\pm}w^{\pm}_{\alpha} - h\langle F^{\alpha}w_{\alpha}\rangle + \langle D^{\alpha\beta}_{\angle}w_{\beta,\zeta}(\phi_{\alpha,\zeta}\cdot\gamma + \tilde{R}_{\alpha,\zeta} + h\tilde{L}_{c\alpha} : A^c_{,\nu}) - hD^{\alpha\beta\gamma\delta}_{\parallel}w_{\alpha}(A_{\gamma\delta,\beta} + \zeta K_{\gamma\delta,\beta})\rangle \tag{80}$$

The integration by parts takes place in the last term here.

From the fact that $w_\alpha(\zeta)$, γ, A and K are arbitrary and taking into account the constraints in eqn (62) one can derive

$$-(D^{\alpha\mu}_{\angle}\phi^{\beta}_{\mu,\zeta})_{,\zeta} = \psi^{\alpha}_{\mu}G^{\mu\beta}, \quad (D^{\alpha\mu}_{\angle}\phi^{\beta}_{\mu,\zeta})\Big|_{\zeta=\pm 1/2} = 0 \tag{81}$$

$$-(D^{\alpha\mu}_{\angle}\tilde{L}^{\beta\gamma\delta}_{c\mu})_{,\zeta} - (D^{\alpha\beta\gamma\delta}_{\parallel} - \langle D^{\alpha\beta\gamma\delta}_{\parallel}\rangle) = \psi^{\alpha}_{\mu}\lambda^{\mu\beta\gamma\delta}_{c}, \quad (D^{\alpha\mu}_{\angle}\tilde{L}^{\beta\gamma\delta}_{c\mu})\Big|_{\zeta=\pm 1/2} = 0 \tag{82}$$

where G and λ_c are Lagrange multipliers that enforce the constraint $\langle \psi^{\beta}_{\alpha}w_{\beta}\rangle = 0$, eqn (62). Equation (81) together with eqns (63) and (64) can be considered as equations for $\phi(\zeta)$ and G.

As soon as $\phi(\zeta)$ is known, the solution of eqn (82) can be written as

$$\tilde{L}^{\beta}_{c\alpha}(\zeta) = \bar{L}^{\beta}_{c\alpha}(\zeta) + \phi^{\eta}_{\alpha,\zeta}(\zeta)G^{-1}_{\eta\mu}\lambda^{\mu\beta}_{c} \tag{83}$$

where $\bar{L}_c$ are the solutions of eqn (82) with zero λ_c that is the same as eqns (54) and (55).

To complete the analysis of these equations, let us notice that if $\tilde{L}_c$ are known, $\tilde{H}_c$ from eqn (70) are given by

$$\tilde{H}^{\beta}_{c\alpha}(\zeta) = \bar{H}^{\beta}_{c\alpha}(\zeta)+\phi^{\eta}_{\alpha}(\zeta)G^{-1}_{\eta\mu}\lambda^{\mu\beta}_{c}+H^{0\beta}_{c\alpha}-\zeta\delta^{\beta}_{\alpha}D^{0}_{c} \tag{84}$$

where $\bar{H}_c$ is the solution of

$$\bar{H}_{c,\zeta} = \bar{L}_c - I\bar{D}_c \quad \langle\bar{H}_c\rangle = 0 \tag{85}$$

that is the same as eqn (50). Here, $\bar{D}_c$ and D^0_c come from eqn (76) and H^0_c is arbitrary constant to be determined later.

Equations for $\tilde{R}_\alpha(\zeta)$ can also be obtained from eqn (80)

$$-(D^{\alpha\beta}_{L}\tilde{R}_{\beta,\zeta})_{,\zeta}-(hF^{\alpha}-f^{\alpha}) = \psi^{\alpha}_{\mu}\lambda^{\mu}_{f}(D^{\alpha\beta}_{L}\tilde{R}_{\beta,\zeta})\Big|_{\zeta=\pm1/2} = \pm P^{\alpha}_{\pm} \tag{86}$$

where λ_f are the Lagrange multipliers that enforce the constraints $\langle\psi^{\beta}_{\alpha}w_{\beta}\rangle = 0$, eqn (62).

The solution of eqn (86) is

$$\tilde{R}_\alpha(\zeta) = \bar{R}_\alpha(\zeta)+\phi^{\eta}_{\alpha}(\zeta)G^{-1}_{\eta\mu}\lambda^{\mu}_{f}+R^{0}_{\alpha} \quad \langle\bar{R}_\alpha\rangle = 0 \tag{87}$$

where $\bar{R}_\alpha$ is the solution of eqn (86) with $\lambda_f = 0$, which is the same as eqn (52) and $R^0_\alpha \equiv \langle\tilde{R}_\alpha(\zeta)\rangle$ is arbitrary constant, to be determined later.

By premultiplying the eqn (86) with ϕ and integration through the thickness one can derive that

$$\langle\psi^{\beta}_{\alpha}\tilde{R}_{\beta}\rangle = G^{-1}_{\alpha\mu}(\lambda^{\mu}_{f}+\phi^{\pm\mu}_{\nu}P^{\nu}_{\pm}+\langle\phi^{\mu}_{\nu}hF^{\nu}\rangle). \tag{88}$$

In the above equations all the quantities with bars do not depend on choice of $\psi(\zeta)$, and expressions for D_c, $\tilde{L}_c$ and $\tilde{H}_c$ contain 54 arbitrary constants:

— $6 = 2\times3$ of D^0_c (each D^0_c is a 2×2 symmetric matrix)
—$24 = 2\times12$ of H^0_c (each $H^{0\beta\gamma\delta}_{c\mu}$ is symmetrical with respect to γ and δ)
—$24 = 2\times12$ of λ_c (each $\lambda^{\nu\beta\gamma\delta}_{c}$ is symmetrical with respect to γ and δ).

If eqns (75), (77)–(79), (81), (82), and (86) are satisfied, the energy eqn (71) reduces to

$$\begin{aligned} J = {} & E_{cl}+P_R-m^{\alpha}\gamma_\alpha-P^{\alpha}_{\pm}(h\phi_\alpha\cdot\gamma+h^2\tilde{H}^{\nu}_{c\alpha}:A^{c}_{,\nu})^{\pm}-\langle hF^{\alpha}(h\phi_\alpha\cdot\gamma+h^2\tilde{H}^{\nu}_{c\alpha}:A^{c}_{,\nu})\rangle \\ & -P^{3}_{\pm}(hD_c:A^c)^{\pm}-\langle hF^{3}(hD_c:A^c)\rangle+h\langle\tfrac{1}{2}D^{\alpha\beta}_{L}(\phi^{\gamma}_{\alpha,\zeta}\gamma_\gamma)(\phi_{\beta,\zeta}\cdot\gamma) \\ & +D^{\alpha\beta}_{L}(\phi^{\gamma}_{\alpha,\zeta}\gamma_\gamma)(h\tilde{L}^{\nu}_{c\beta}:A^{c}_{,\nu}+\tilde{R}_{\beta,\zeta})+\tfrac{1}{2}D^{\alpha\beta}_{L}(h\tilde{L}^{\nu}_{c\alpha}:A^{c}_{,\nu}+\tilde{R}_{\alpha,\zeta})(\alpha\to\beta) \\ & -hD^{\alpha\beta\gamma\delta}_{\parallel}(A_{\gamma\delta,\beta}+\zeta K_{\gamma\delta,\beta})\times(\phi_\alpha\cdot\gamma-\zeta\gamma_\alpha+\tilde{R}_\alpha+h\tilde{H}^{\nu}_{c\alpha}:A^{c}_{,\nu})\rangle. \end{aligned} \tag{89}$$

The term with $D_\parallel$ is obtained through the integration by parts with respect to x^α here. One can obtain that

$$\langle D^{\alpha\beta}_{L}(\phi^{\mu}_{\alpha,\zeta}\gamma_\mu)(\phi^{\nu}_{\beta,\zeta}\gamma_\nu)\rangle = G^{\mu\nu}\gamma_\mu\gamma_\nu \tag{90}$$

and it becomes clear that G is the transverse shear stiffness matrix.

Let us collect terms in eqn (89) that represent the interaction between γ and $A_{,\nu}$, $K_{,\nu}$ and external forces

$$\begin{aligned} & -P^{\alpha}_{\pm}(h\phi_\alpha\cdot\gamma)^{\pm}-\langle hF^{\alpha}(h\phi_\alpha\cdot\gamma)\rangle+m^{\alpha}\gamma_\alpha h\langle D^{\alpha\mu}_{L}(\phi^{\nu}_{\alpha,\zeta}\gamma_\nu)(h\tilde{L}^{\beta\gamma\delta}_{c\mu}A^{c}_{\gamma\delta,\beta}+\tilde{R}_{\mu,\zeta}) \\ & \qquad -hD^{\alpha\beta\gamma\delta}_{\parallel}(A_{\gamma\delta,\beta}+\zeta K_{\gamma\delta,\beta})(\phi_\alpha\cdot\gamma-\zeta\gamma_\alpha)\rangle \end{aligned} \tag{91}$$

After integration by parts in the second line, and taking into account eqns (82), (86), and (91), can be transformed to

$$h\left\langle h(\psi_\mu^\alpha\lambda_e^{\mu\beta}+\langle\mathscr{I}_e D^{\alpha\beta}\rangle):A_{,\beta}^e\phi_\alpha\cdot\gamma+h\zeta D_\parallel^{\alpha\beta}:(A_{,\beta}+\zeta K_{,\beta})\gamma_\alpha+\gamma\cdot\phi_\alpha(\psi_\mu^\alpha\lambda_f^\mu-f^\alpha)+\frac{1}{h}m^\alpha\gamma_\alpha\right\rangle. \tag{92}$$

Taking into account eqns (64), one can see that choosing

$$\lambda_e^{\alpha\beta\gamma\delta}=-\langle\zeta\mathscr{I}_e D_\parallel^{\alpha\beta\gamma\delta}\rangle \tag{93}$$

and

$$\lambda_f^\alpha=-\frac{1}{h}m^\alpha \tag{94}$$

make eqn (92) equal to zero. It is important to note that λ_e and λ_f, given by eqns (93) and (94), respectively, do not depend on one's choice of $\psi(\zeta)$.

Therefore, we get a further reduction of eqn (89) given by

$$\begin{aligned}J=E_{cl}+hG^{\alpha\beta}\gamma_\alpha\gamma_\beta+P_R-P_\pm^\alpha(h^2\tilde{H}_{e\alpha}^\nu:A_{,\nu}^e)^\pm-\langle hF^\alpha(h^2\tilde{H}_{e\alpha}^\nu:A_{,\nu}^e)\rangle\\ -P_\pm^3(hD_e:A^e)^\pm-\langle hF^3(hD_e:A^e)\rangle+h\langle\tfrac{1}{2}D_\perp^{\alpha\beta}(h\tilde{L}_{e\alpha}^\nu:A_{,\nu}^e+\tilde{R}_{\alpha,\zeta})(\alpha\to\beta)\\ -hD_\parallel^{\alpha\beta\gamma\delta}(A_{\gamma\delta,\beta}+\zeta K_{\gamma\delta,\beta})(\tilde{R}_\alpha+h\tilde{H}_{e\alpha}^\nu:A_{,\nu}^e)\rangle.\end{aligned} \tag{95}$$

Let us collect terms that contain external forces

$$\begin{aligned}h^2\langle D_\perp^{\alpha\beta}(\tilde{L}_{e\alpha}^\nu:A_{,\nu}^e)(\tilde{R}_{\beta,\zeta})-D_\parallel^{\alpha\beta\gamma\delta}(A_{\gamma\delta,\beta}+\zeta K_{\gamma\delta,\beta})(\tilde{R}_\alpha)\rangle-P_\pm^\alpha(h^2\tilde{H}_{e\alpha}^\nu:A_{,\nu}^e)^\pm-\langle hF^\alpha(h^2\tilde{H}_{e\alpha}^\nu:A_{,\nu}^e)\rangle\\ -P_\pm^3(hD_e:A^e)^\pm-\langle hF^3(hD_e:A^e)\rangle.\end{aligned} \tag{96}$$

Here, we dropped the quadratic term, $\tilde{R}\cdot D_\perp\cdot\tilde{R}$, which depends on forces only and cannot be varied.

Integrating by parts with respect to ζ in the first line and using eqn (82), one can reduce this expression to

$$\begin{aligned}h^2[\langle\tilde{R}_\alpha\psi_\mu^\alpha\rangle(\lambda_e^{\mu\beta}:A_{,\beta}+\lambda_b^{\mu\beta}:K_{,\beta})+\langle\tilde{R}_\alpha\rangle(\langle D_\parallel^{\alpha\beta}\rangle:A_{,\beta}+\langle\zeta D_\parallel^{\alpha\beta}\rangle:K_{,\beta})]\\ -P_\pm^\alpha(h^2\tilde{H}_{e\alpha}^\nu:A_{,\nu}^e)^\pm-\langle h^3F^\alpha\tilde{H}_{e\alpha}^\nu:A_{,\nu}^e\rangle-P_\pm^3(hD_eA^e)^\pm-\langle hF^3(hD_eA^e)\rangle.\end{aligned} \tag{97}$$

Let us substitute here eqns (84), (87) and (88) yielding

$$\begin{aligned}&h^2G_{\alpha\mu}^{-1}(\lambda_f^\mu+\phi_\nu^{\pm\mu}P_\pm^\nu+\langle\phi_\nu^\mu hF^\nu\rangle)(\lambda_e^{\mu\beta}:A_{,\beta}^e)+h^2\langle\tilde{R}_\alpha\rangle(\langle D_\parallel^{\alpha\beta}\rangle:A_{,\beta}+\langle\zeta D_\parallel^{\alpha\beta}\rangle:K_{,\beta})\\ &-h^2P_\pm^\alpha(\bar{H}_{e\alpha}^\nu:A_{,\nu}+\bar{H}_{b\alpha}^\nu:K_{,\nu})^\pm-h^2P_\pm^\alpha\phi_\alpha^{\pm\eta}(\zeta)G_{\eta\mu}^{-1}(\lambda_e^{\mu\nu}:A_{,\nu}+\lambda_b^{\mu\nu}:K_{,\nu})\\ &-h^2P_\pm^\alpha(H_{e\alpha}^{0\beta}-\zeta\delta_\alpha^\nu D_e^0):A_{,\nu}-h^2P_\pm^\alpha(H_{b\alpha}^{0\nu}-\zeta\delta_\alpha^\nu D_b^0):K_{,\nu}-h^2\langle hF^\alpha(\bar{H}_{e\alpha}^\nu:A_{,\nu}+\bar{H}_{b\alpha}^\nu:K_{,\nu})\rangle\\ &-h^2\langle hF^\alpha\phi_\alpha^{\pm\eta}(\zeta)G_{\eta\mu}^{-1}\rangle(\lambda_e^{\mu\beta}:A_{,\beta}+\lambda_b^{\mu\beta}:K_{,\beta})-\langle hF^\alpha(H_{e\alpha}^{0\nu}-\zeta\delta_\alpha^\nu D_e^0):A_{,\nu}\rangle\\ &-\langle hF^\alpha(H_{b\alpha}^{0\nu}-\zeta\delta_\alpha^\nu D_b^0):K_{,\nu}\rangle-P_\pm^3(hD_e:A^e)^\pm-\langle hF^3(hD_e:A^e)\rangle.\end{aligned} \tag{98}$$

One can see that terms with ϕ are cancelled out. After integrating by parts with respect to x_α and collecting all the terms depending on external forces, we obtain

$$P_a = -P^3_{\pm} h D^{\pm}_c - h^2 \langle F^3 D_c \rangle + h^2 P^{\alpha}_{-,\nu} \bar{H}^{-\nu}_{a\alpha} + h^3 \langle F^{\alpha}_{,\nu} \bar{H}^{\nu}_{a\alpha} \rangle + R^0_{\alpha,\beta} \langle \mathcal{I}_c D^{\alpha\beta} \rangle \tag{99}$$

which is the same as eqn (46) after setting the constant R^0 equal to zero.

We are left with the part 3 of the 2-D energy, which needs to be made zero, given as

$$E_3 = \langle \tfrac{1}{2} D^{\alpha\beta}_{\perp} (\bar{L}^{\nu}_{c\alpha} : A^{c}_{,\nu})(\alpha \to \beta) - D^{\alpha\beta;\delta}_{\parallel} (A_{\gamma\delta,\beta} + \zeta K_{\gamma\delta,\beta})(\bar{H}^{\nu}_{c\alpha} : A^{c}_{,\nu}) \rangle. \tag{100}$$

This is a quadratic form of 12 variables, $A_{,\nu}$ and $K_{,\nu}$. It contains 78 coefficients which are functionals of $\psi(\zeta)$ and $30 \equiv 54-24$ constants so far. Those 78 coefficients need to be made zero.

One can substitute eqn (83) and (84) into eqn (100)

$$\begin{aligned} E_3 = \langle \tfrac{1}{2} D^{\alpha\beta}_{\perp} [\bar{L}^{\nu}_{e\alpha} : A_{,\nu} + \bar{L}^{\nu}_{b\alpha} : K_{,\nu} + \phi^{\eta}_{\alpha,\zeta} G^{-1}_{\eta\mu} (\lambda^{\mu\nu}_{e} : A_{,\nu} + \lambda^{\mu\nu}_{b} : K_{,\nu})][\alpha \to \beta] \\ - D^{\alpha\beta;\delta}_{\parallel} (A_{\gamma\delta,\beta} + \zeta K_{\gamma\delta,\beta})[\bar{H}^{\nu}_{e\alpha} : A_{,\nu} + \bar{H}^{\nu}_{b\alpha} : K_{,\nu} + \phi^{\eta}_{\alpha} G^{-1}_{\eta\mu} (\lambda^{\mu\nu}_{e} : A_{,\nu} + \lambda^{\mu\nu}_{b} : K_{,\nu}) \\ + (H^{0\nu}_{e\alpha} - \zeta\delta^{\nu}_{\alpha} D^{0}_{e}) : A_{,\alpha} + (H^{0\nu}_{b\alpha} - \zeta\delta^{\nu}_{\alpha} D^{0}_{b}) : K_{,\alpha}] \rangle. \end{aligned} \tag{101}$$

Collecting terms that contain ϕ one gets

$$\langle D^{\alpha\beta}_{\perp} (\bar{L}^{\nu}_{e\alpha} : A_{,\nu} + \bar{L}^{\nu}_{b\alpha} : K_{,\nu}) \phi^{\eta}_{\alpha,\zeta} - D^{\alpha\beta}_{\parallel} : (A_{,\beta} + \zeta K_{,\beta}) \phi^{\eta}_{\alpha} \rangle G^{-1}_{\eta\mu} (\lambda^{\mu\nu}_{e} : A_{,\nu} + \lambda^{\mu\nu}_{b} : K_{,\nu}).$$

Again, integrating by parts with respect to ζ and taking into account eqn (82) for $\bar{L}_c$, one can see that this is equal to zero, and we are left with

$$\begin{aligned} E_3 = \langle \tfrac{1}{2} D^{\alpha\beta}_{\perp} (\bar{L}^{\nu}_{e\alpha} : A_{,\nu} + \bar{L}^{\nu}_{b\alpha} : K_{,\nu})(\alpha \to \beta) + \tfrac{1}{2} G^{-1}_{\alpha\beta} (\lambda^{\alpha\nu}_{e} : A_{,\nu} + \lambda^{\alpha\nu}_{b} : K_{,\nu})(\alpha \to \beta) \\ - D^{\alpha\beta}_{\parallel} : (A_{,\beta} + \zeta K_{,\beta})(\bar{H}^{\nu}_{e\alpha} : A_{,\nu} + \bar{H}^{\nu}_{b\alpha} : K_{,\nu} + (H^{0\nu}_{e\alpha} - \zeta\delta^{\nu}_{\alpha} D^{0}_{e}) : A_{,\alpha} + (H^{0\nu}_{b\alpha} - \zeta\delta^{\nu}_{\alpha} D^{0}_{b}) : K_{,\alpha}) \rangle. \end{aligned} \tag{102}$$

This can be rewritten in the form

$$E_3 = \tfrac{1}{2} A_{,\mu} : E^{\mu\nu}_{3e} : A_{,\nu} + \tfrac{1}{2} K_{,\mu} : E^{\mu\nu}_{3b} : K_{,\nu} + A_{,\mu} : E^{\mu\nu}_{3eb} : K_{,\nu} \tag{103}$$

with

$$\begin{aligned} E^{\mu\nu\alpha\beta;\delta}_{3e} &= \langle \bar{L}^{\mu\alpha\beta}_{e\eta} D^{\eta\theta}_{\perp} \bar{L}^{\nu\gamma\delta}_{e\theta} \rangle + \lambda^{\eta\mu\alpha\beta}_{e} G^{-1}_{\eta\theta} \lambda^{\theta\nu\gamma\delta}_{e} - 2 \langle (\bar{H}^{\mu\alpha\beta}_{e\eta} + H^{0\mu\alpha\beta}_{e\eta} - \zeta\delta^{\mu}_{\eta} D^{0\alpha\beta}_{e}) D^{\eta\nu\gamma\delta}_{\parallel} \rangle \\ E^{\mu\nu\alpha\beta;\delta}_{3b} &= \langle \bar{L}^{\mu\alpha\beta}_{b\eta} D^{\eta\theta}_{\perp} \bar{L}^{\nu\gamma\delta}_{b\theta} \rangle + \lambda^{\eta\mu\alpha\beta}_{b} G^{-1}_{\eta\theta} \lambda^{\theta\nu\gamma\delta}_{b} - 2 \langle \zeta (\bar{H}^{\mu\alpha\beta}_{b\eta} + H^{0\mu\alpha\beta}_{b\eta} - \zeta\delta^{\mu}_{\eta} D^{0\alpha\beta}_{b}) D^{\eta\nu\gamma\delta}_{\parallel} \rangle \\ E^{\mu\nu\alpha\beta;\delta}_{3eb} &= \langle \bar{L}^{\mu\alpha\beta}_{e\eta} D^{\eta\theta}_{\perp} \bar{L}^{\nu\gamma\delta}_{b\theta} \rangle + \lambda^{\eta\mu\alpha\beta}_{e} G^{-1}_{\eta\theta} \lambda^{\theta\nu\gamma\delta}_{b} - \langle (\bar{H}^{\mu\gamma\delta}_{b\eta} + H^{0\mu\gamma\delta}_{b\eta} - \zeta\delta^{\mu}_{\eta} D^{0\gamma\delta}_{b}) D^{\eta\nu\alpha\beta}_{\parallel} \rangle \\ &\quad - \langle \zeta (\bar{H}^{\mu\alpha\beta}_{e\eta} + H^{0\mu\alpha\beta}_{e\eta} - \zeta\delta^{\mu}_{\eta} D^{0\alpha\beta}_{e}) D^{\eta\nu\gamma\delta}_{\parallel} \rangle. \end{aligned} \tag{104}$$

One can see that E_3 depends on the choice of ψ only via G^{-1}. Thus, the 78 coefficients of E_3 depend on 33 variables, and it becomes clear that the system is overdetermined in the general case.

Recovering relations for u_α can now be written as

$$\begin{aligned} u_\alpha = v_\alpha - h\zeta v_{3,\alpha} + h^2 \bar{H}^{\nu}_{e\alpha} : A_{,\nu} + h^2 \bar{H}^{\nu}_{b\alpha} : K_{,\nu} + h\bar{R}_\alpha + h\phi^{\eta}_{\alpha} [\gamma_\eta + G^{-1}_{\eta\mu} (h\lambda^{\mu\nu}_{e} : A_{,\nu} + h\lambda^{\mu\nu}_{b} : K_{,\nu} + \lambda^{\mu}_{f})] \\ + h^2 H^{0\nu}_{e\alpha} : A_{,\nu} + h^2 H^{0\nu}_{b\alpha} : K_{,\nu} - h^2 \zeta D^{0}_{e} : A_{,\alpha} - h^2 \zeta D^{0}_{b} : K_{,\alpha} + w_\alpha. \end{aligned} \tag{105}$$

Using eqns (33), (35), (36), (93), and (94), the coefficient of ϕ can be transformed as follows

$$h\gamma_{\eta}+hG_{\eta\mu}^{-1}(h\lambda_{e}^{\mu\nu}:A_{,\nu}+h\lambda_{b}^{\mu\nu}:K_{,\nu}+\lambda_{f}^{\mu}) = G_{\eta\mu}^{-1}hP_{b,\nu}^{\mu\nu} \sim h\left(\frac{h}{l}\right)^{3}\varepsilon.$$

This means that the term with ϕ is a higher-order term and can be neglected. The analogous statement is true for the u_3 final expansion, and the recovering relations take their final form of eqn (47).

Note, that explicit definition of 2-D variables now is

$$\langle u_{\alpha}\rangle = v_{\alpha}+h^{2}H_{e\alpha}^{0\nu}:A_{,\nu}+h^{2}H_{b\alpha}^{0\nu}:K_{,\nu}$$

$$\langle u_{3}\rangle = v_{3}+hD_{e}^{0}:A+hD_{b}^{0}:K. \tag{106}$$

It is different from what is suggested by the main exchange rules, eqn (59–64), by high order terms. Application of eqn (106) never occurs in practice, and it is not important.

This completes the derivation.

10. NUMERICAL RESULTS

In this section we give a few sample results to illustrate the power of the theory. For our numerical results, we choose a fiber-reinforced composite material which has the following material properties

$$E_L = 25\times 10^6 \text{ psi} \qquad E_T = 10^6 \text{ psi}$$

$$G_{LT} = 0.5\times 10^6 \text{ psi} \quad G_{TT} = 0.2\times 10^6 \text{ psi}$$

$$\nu_{LT} = \nu_{TT} = 0.25$$

where L signifies the direction parallel to the fibers and T the transverse direction. These properties, along with a ply angle, allow the calculation of the material matrix. We consider [15°] as an example 1-layer plate and [15°, −15°] as an example 2-layer layup. The test problem is the bending of a simply-supported infinite plate under a sinusoidal load equally distributed on the upper and lower surfaces. This problem has a known exact solution by Pagano (1969, 1970) which can be compared with results from the present theory as well as from NCPT.

To validate our approach we have expanded the exact solution in powers of the small parameter h/l, where h is the thickness of the plate and l is the characteristic wavelength of the deformation. The expansion is compared with analogous solutions from CPT, NCPT and the present theory. For the 1-layer case, the expansions of NCPT and the present theory are quite close to each other and to that of the exact solution. In this case the present theory agrees with the result from Atilgan and Hodges (1992), which is claimed to be asymptotically correct for homogeneous plates. However, NCPT is not asymptotically correct.

The results of the expansion for the displacement fields of the 2-layer case are represented in Table 1. The classical theory, by which we mean the above described theory, is supposed to provide the correct result for the leading term, and it indeed does. It can be seen, however, that each coefficient of the new theory is closer to the corresponding one from the exact expansion than the corresponding coefficient of NCPT. Figures 1–3 compare the results graphically. Figure 1 shows the distributions of the displacement through the thickness for the 2-layer plate. Figure 2a and b give the comparison of the relative error for the average displacement and the 3-D displacement with the average subtracted out, respectively. These results show clearly the power of the present approach. Not only is the average displacement much more accurate than that of CPT and NCPT, but the improvement in accuracy of the 3-D displacement field relative to the other approaches is even greater.

Figure 3 shows transverse strain calculated directly from the recovering relations and transverse stress from the strains and the 3-D constitutive equations. Such excellent

Table 1. Asymptotical expansion of the normalized displacement for an infinite two-layer plate with layup [15°, −15°] where $\zeta \in [-\frac{1}{2}, \frac{1}{2}]$ is through-the-thickness coordinate and h/l is the thickness parameter—the quotient of thickness by the characteristic wavelength of the load. The upper/lower sign corresponds to the first/second layer. The function is normalized to have the first term independent of h/l

	Lateral displacement
Classical	-117ζ
NCPT	$-117\zeta+(-68.7\zeta-440\zeta^3)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$
Present	$-117\zeta+(34.3\zeta\pm444\zeta^2-858\zeta^3)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$
Exact	$-117\zeta+(30.5\zeta\pm444\zeta^2-858\zeta^3)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$
	Longitudinal displacement
Classical	-82.6
NCPT	$-82.6+(-21.7\mp112\zeta\pm150\zeta^3)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$
Present	$-82.6+(-3.44\mp69.5\zeta-219\zeta^2\mp199\zeta^3)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$
Exact	$-82.6+(1.56\mp69.5\zeta-219\zeta^2\mp199\zeta^3)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$
	Transverse displacement
Classical	$117+(12\mp12.4\zeta-181\zeta^2)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$
NCPT	$117+(263\mp12.4\zeta-181\zeta^2)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$
Present	$117+(238\mp12.4\zeta-181\zeta^2)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$
Exact	$117+(234\mp12.4\zeta-181\zeta^2)\left(\frac{h}{l}\right)^2+O\left(\frac{h}{l}\right)^3$

agreement with the exact solution has not been obtained with any first-order shear deformation theory within the authors' knowledge. Indeed, it is even better than that of many higher-order and layerwise theories. This is especially remarkable considering that through-the-thickness integration of the 3-D equilibrium equations was not required to get this agreement!

11. CONCLUDING REMARKS

The variational-asymptotical method is applied to the development of anisotropic plate theory from 3-D elasticity. A complete Reissner-like 2-D plate theory is developed for orthotropic plates which is as close to asymptotical correctness as can be obtained. In the course of developing this theory, several new observations were made:

—the conditions for any plate or shell theory to be asymptotically correct are obtained. These conditions require that certain coefficients in the full 3-D energy, as expressed in terms the 2-D variables, vanish;
—an asymptotically correct form of the theory for homogeneous plates is presented;

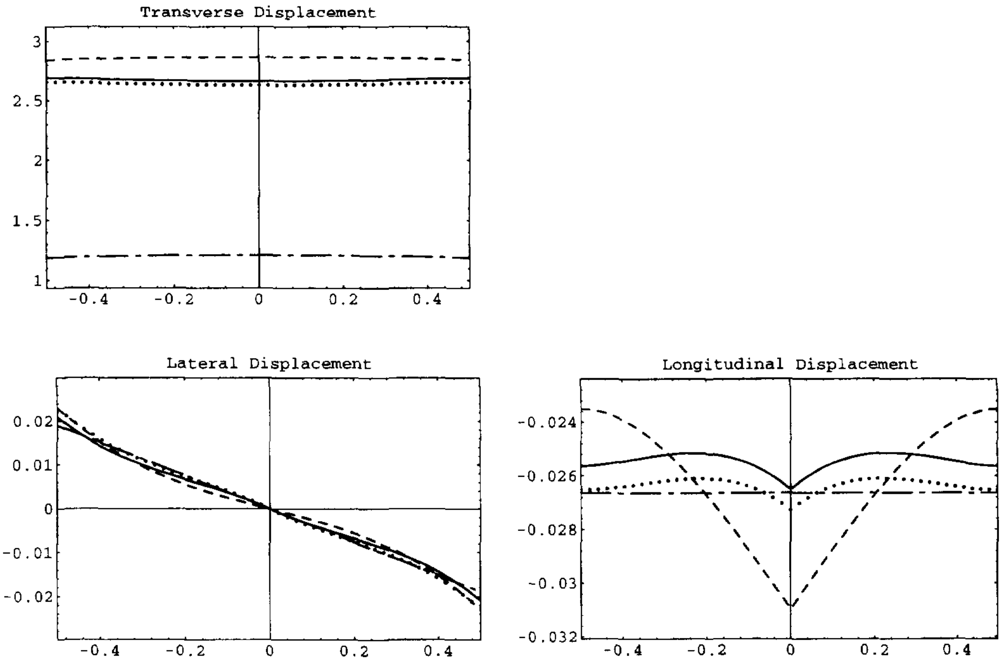
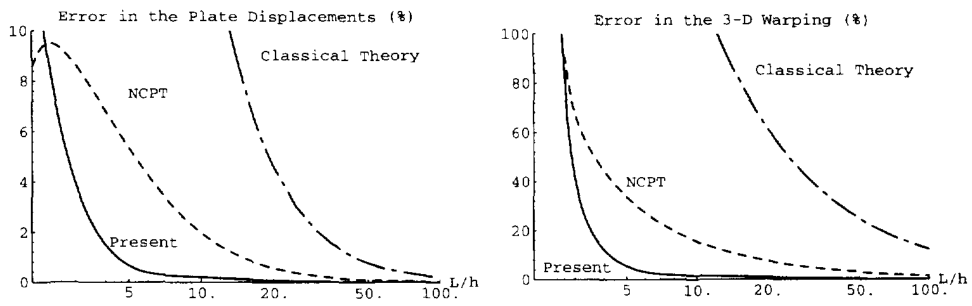


Fig. 1. Distribution of the displacement components vs the through-the-thickness coordinate from the kinematic recovery relations for an infinite two-layer plate with layup [15°, −15°]. The quotient of the thickness of the plate by its width is equal to 0.25.

—for the most general inhomogeneity of plates, an asymptotically correct Reissner-like theory may not exist. However, it is shown that a minimization procedure can always be applied to bring such a theory as close as possible to asymptotical correctness. That minimization has been carried out in a subspace of the full space of admissible values of quantities to be found and is implemented in Mathematica (Wolfram (1991)) code;

—numerical results obtained from the present theory, including both 2-D and 3-D quantities for a variety of particular cases of laminated plates, are presented. The accuracy of these results is indeed superior to that of any extant first-order shear deformation theory known to authors. Indeed, the theory gives accurate transverse stresses and strains without employing integration of 3-D equilibrium equations. It should be noted that results for several different laminates (both symmetric and cross-ply) were examined in the course of the work, and the results obtained for all exhibited accuracy comparable to those presented.



a. $\frac{\|\langle \vec{u}_R\rangle - \langle \vec{u}\rangle\|}{\|\langle \vec{u}\rangle\|} 100\%$ b. $\frac{\max_\zeta \|\vec{u}_R - \langle \vec{u}\rangle\|}{\max_\zeta \|\vec{u} - \langle \vec{u}\rangle\|} 100\%$

Fig. 2. Relative displacement errors vs the thickness parameter—the quotient of the width of the plate by its thickness. The horizontal axis is logarithmic. $\mathbf{u}$ is the 3-D exact vector displacement, $\mathbf{u}_R$ is the 3-D vector displacement obtained by the recovery relations of a plate theory, $\langle\cdot\rangle$ is a through-the-thickness average.

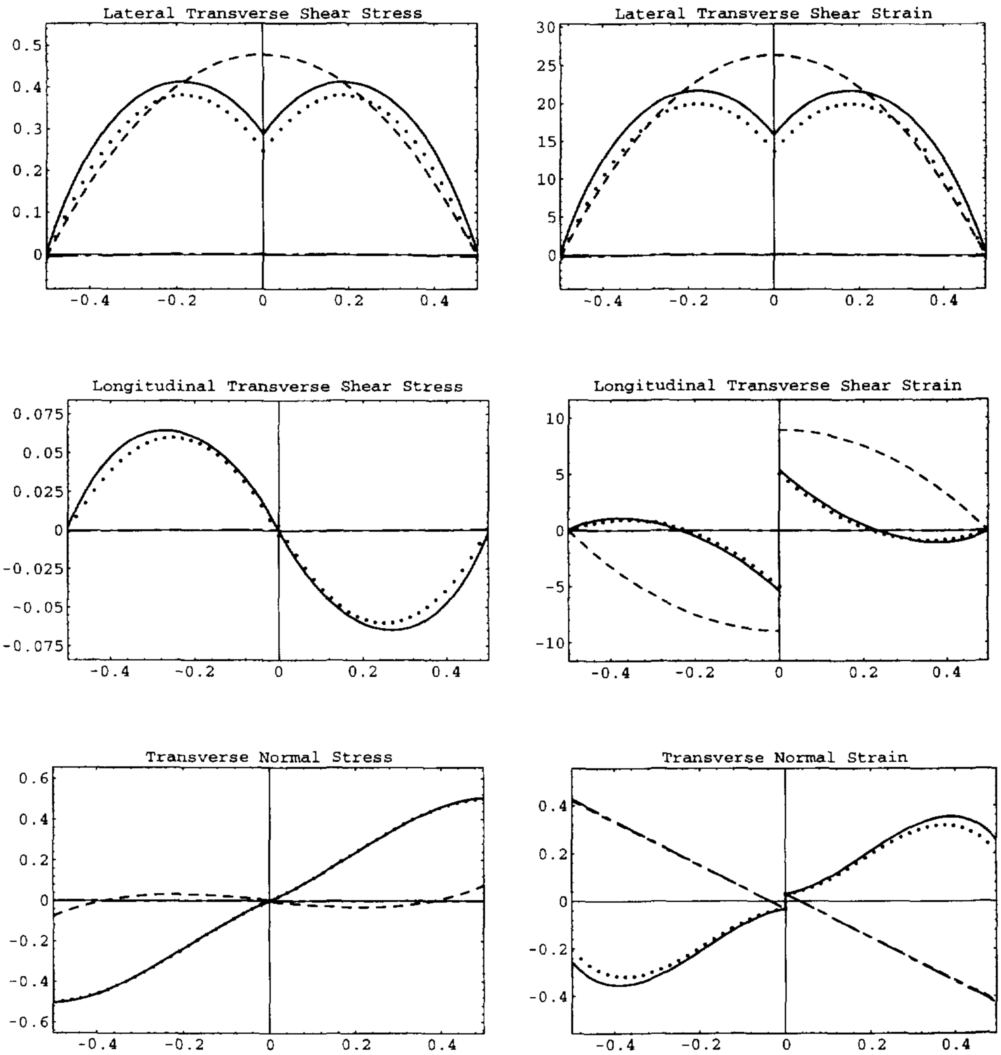


Fig. 3. Distribution of transverse stresses and strains versus the through-the-thickness coordinate for an infinite two-layer plate with layup [15°, −15°]. The strains were obtained using the kinematic recovery relations, and stresses are obtained from the 3-D constitutive equations. The quotient of the thickness of the plate by its width is equal to 0.25. The legend is the same as on the previous figures.

Future work to be done includes development of a nonlinear theory, extension to shells, and application of this methodology to creation of a new generation of simple and accurate finite elements. Although the development herein may appear complex, the implementation of the theory is really quite straightforward in the context of symbolic manipulation software. The development of the nonlinear theory is expected to be of comparable difficulty.

Acknowledgments—This work was supported, in part, by the U.S. Army Research Office through the Center of Excellence for Rotorcraft Technology, the technical monitor of which is Dr Gary L. Anderson. Technical discussions with Prof. Victor L. Berdichevsky are gratefully acknowledged.

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